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SOME LOW ENERGY NEUTRINO CROSS SECTIONS

by

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ABSTRACT

The cross sections for elastic neutrino and antineutrino scattering on electrons, and antineutrino absorption on protons (giving a neutron and a positron), are calculated using the V-A four-fermion interaction. Recoil electron spectra are presented for the elastic scatterings, and the angular distribution of positrons is given for the absorption process, in addition to total cross sections for all these processes. Results are given analytically and in graphical form.

The steps necessary to perform these calculations are explained in detail.

For the elastic scatterings the results are expected to be valid for center of mass energies much less than the mass of the intermediate boson (if it exists), and also for energies too small to probe the structure of the neutrino and/or electron (if there is any). Therefore the formulae should be reliable up to at least 50 MeV neutrinos and antineutrinos. For the absorption process the anomalous magnetic moment of the nucleons presents the first correction (with increasing energy) to the calculation presented here which is accurate only below 10 MeV antineutrino energy.

An Appendix reproduces some lecture notes from 1957 on nuclear beta decay. Although a parity-conserving interaction was used in those notes, the formulae are still valid for all those results involving initially unpolarized nuclei and in which all final polarizations are summed over.

CONTENTS

I. Introduction

Calculations of cross sections involving neutrinos have appeared in the literature, and none of the results presented in this report are new although we do point out some errors in the literature. It was felt desirable, however, to collect in a single place the detailed steps involved in performing these calculations.

We use the four-fermion interaction as given by Feynman and Gell-mann[1], slightly generalized to allow for unequal vector and axial vector coupling constants. This interaction does not include "weak magnetism"[2] nor an "induced" pseudoscalar[3] term. The former term for processes involving nucleons becomes increasingly important as the energy increases, and represents, already at 10 MeV, a correction of ~4% to our calculation of $\tilde{\nu} + p \rightarrow n + e^{+}$. The pseudoscalar term is only significant for processes involving μ-mesons. A complete cross section formula containing contributions from all possible Lorentz invariant interactions of the four-fermion type is given in reference 4.

If there is an intermediate boson, then the calculations presented here of neutrino and antineutrino scattering on electrons will be valid only up to center of mass energies which are much smaller than the boson mass.

After a brief section on notation, Section III follows in detail the calculation of the square of the matrix element, summed over all polarizations of all four fermions. The utility of this quantity, with special reference to the role of neutrinos, is discussed. Sections IV and V summarize the needed kinematics and density of states formulae, and in Section VI all the results are collected in a cross section formula. Section VII applies these results to $\tilde{\nu} + e \rightarrow \tilde{\nu} + e$, $\nu + e \rightarrow \nu + e$, and $\tilde{\nu} + p \rightarrow n + e^{+}$, and numerical cross sections are presented in graphical form. Included in this section is a discussion of the possible role of neutral symmetric currents on the elastic scattering processes.

Some lecture notes from 1957 on nuclear beta decay are included as an Appendix since they contain many of the results about projection operators which are needed for the present calculations. All the results in the Appendix which refer to initially unpolarized nuclei, and final states in which all polarizations are summed over, are valid even though a parity conserving interaction was used since no pseudoscalar quantities can be formed from just the two lepton momenta.

II. Notation

Greek indices run from zero to three; Roman indices from one to three. Three-dimensional vectors are written with arrows, $\vec{p}$; four-dimensional vectors with a wavy underline, $\underset{\sim}{p}$. Dot products are defined by $\vec{p}\cdot\vec{q} \equiv \sum_{i=1}^{3} p_i q_i$; $\underset{\sim}{p}\cdot\underset{\sim}{q} \equiv p_o q_o - \sum_{i=1}^{3} p_i q_i = \sum_{\mu=o}^{3} p_\mu q_\mu g_{\mu\mu} = \sum_{\mu,\nu=o}^{3} p_\mu q_\nu g_{\mu\nu}$, where the metric tensor $g_{\mu\nu}$ has been introduced for later convenience:

$$g \equiv \begin{array}{c|cccc} \mu \backslash \nu & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 2 & 0 & 0 & -1 & 0 \\ 3 & 0 & 0 & 0 & -1 \end{array} .$$

The γ-matrices which shall be used are the following:

$$\gamma_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}, \quad i\gamma_5 \equiv i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

where I and $\vec{\sigma}$ are the two-dimensional identity and Pauli spin matrices, respectively. Some properties of these matrices are that

$$\gamma_0^\dagger = \gamma_0; \; \vec{\gamma}^\dagger = -\vec{\gamma}; \; \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}; \; \gamma_\nu^\dagger\gamma_0 = \gamma_0\gamma_\nu; \; \gamma_5\gamma_\mu = -\gamma_\mu\gamma_5; \; (i\gamma_5)^2 = 1.$$

The dagger superscript denotes the Hermitean conjugate.

$\hbar$ and c will often be set equal to unity, but all cross sections will finally be expressed in cm^2. The symbol ν or $\tilde{\nu}$ without a subscript refers to the electron's neutrino or antineutrino; the μ-meson's neutrino is designated ν_μ.

III. Interaction

We shall use a slightly generalized version of the Feynman, Gell-Mann interaction[1] which allows for unequal vector and axial-vector strengths. This is not the most general possible Lorentz invariant interaction[4], but is valid at low momentum transfers.

$$H = \frac{G}{\sqrt{2}} \sum_{\mu=0}^{3} \left[\bar{\psi}_A \gamma_\mu (1 - i\,\gamma_5)\, \psi_B\right]\left[\bar{\psi}_C \gamma_\mu\, (f - gi\gamma_5)\, \psi_D\right] g_{\mu\mu} \quad .$$

A and B are leptons, but C and D need not be. The ψ's are the field operators (containing spinors) for the corresponding particles, with $\bar{\psi} = \psi^\dagger \gamma_0$. $G = 1.41 \times 10^{-49}$ erg-cm^3 $= 1.01 \times 10^{-5}\ (\hbar c)^3 / \left(M_p c^2\right)^2$ with M_p the mass of the proton.

In first order perturbation theory, the matrix element M of this Hamiltonian looks just like H except that the ψ's are now just the spinors appropriate to the particles,

$$M = \frac{G}{\sqrt{2}} \sum_{\mu=0}^{3} \left(\bar{\psi}_A \gamma_\mu a \psi_B\right)\left(\bar{\psi}_C \gamma_\mu b \psi_D\right) g_{\mu\mu} \quad ,$$

where $a \equiv 1 - i\gamma_5$, $b = f - gi\gamma_5$. Note that $a^\dagger = a$, $b^\dagger = b$ (assuming f and g are real), $a^2 = 2a$, $b^2 = 2cd$ where $c \equiv (f^2 + g^2)/2$, and $d \equiv 1 - \frac{fg}{c}\, i\gamma_5$. Also, $(f - gi\gamma_5)(f + gi\gamma_5) = f^2 - g^2$.

The quantity needed to calculate cross sections is

$$|M|^2 = MM^* = \frac{G^2}{2} \sum_{\mu,\nu=0}^{3} \left(\bar{\psi}_A \gamma_\mu a \psi_B\right)\left(\bar{\psi}_A \gamma_\nu a \psi_B\right)^*\left(\bar{\psi}_C \gamma_\mu b \psi_D\right)\left(\bar{\psi}_C \gamma_\nu b \psi_D\right)^* g_{\mu\mu} g_{\nu\nu} \quad .$$

$$\left(\bar{\psi}_A \gamma_\nu a \psi_B\right)^* = \psi_B^* a^\dagger \gamma_\nu^\dagger \gamma_0^\dagger \psi_A = \psi_B^* a \gamma_0 \gamma_\nu \psi_A = \psi_B^* \gamma_0 \gamma_\nu a \psi_A$$

$$= \bar{\psi}_B \gamma_\nu a \psi_A \quad ,$$

using the properties of the γ-matrices listed previously.

Define

$$(AB)_{\mu\nu} \equiv \psi_A^* \gamma_o \gamma_\mu a \psi_B \psi_B^* \gamma_o \gamma_\nu a \psi_A$$

$$(CD)_{\mu\nu} \equiv \psi_C^* \gamma_o \gamma_\mu b \psi_D \psi_D^* \gamma_o \gamma_\nu b \psi_C \quad ,$$

Then,

$$|M|^2 = \frac{G^2}{2} \sum_{\mu,\nu=0}^{3} (AB)_{\mu\nu} (CD)_{\mu\nu} g_{\mu\mu} g_{\nu\nu} \quad .$$

To evaluate this expression one could insert explicit spinors which tell whether A is a particle or antiparticle and what its momentum and polarization are; do the same for B, C, and D; and carry out the matrix multiplication. This would be the simplest thing to do if all the particle polarizations were actually measured. If some of the initial state particles are unpolarized, then $|M|^2$ must be evaluated separately for each initial polarization state and an <u>average</u> taken. Similarly, if some of the polarizations of the final state particles are not measured, then $|M|^2$ must be calculated separately for each final polarization state and the <u>sum</u> taken. If $|M|^2$ must be recalculated for several different polarizations states, the use of explicit spinors becomes very tedious. An alternative procedure which could be used even if all the polarizations were measured, but which is especially useful if they are not (because it avoids all the work just mentioned), is the use of projection operators which only permit the desired energy states to contribute (positive for particles, negative for antiparticles). Also, if desired, one can pick out particular polarization states. After inserting these projection operators one can sum $|M|^2$ over all four spinors of a given momentum (two different energies and two different polarizations), and this eliminates the spinors via the closure relation[5]. We shall work out only the case where the polarizations are not of interest[6].

A special word about neutrinos is required. The Feynman, Gell-Mann interaction which we are using is such that only left-handed neutrinos (and right-handed antineutrinos) can take part. Neutrinos of the opposite polarization (even if they did exist) would give rise to a zero value for $|M|^2$. Any neutrino which is present in the final state will be completely polarized. There is no harm, however, in formally summing $|M|^2$ over the final polarization states of this neutrino since the "wrong" polarization state automatically contributes zero to the sum. Assuming that any neutrino present in the initial state is completely polarized the "correct" way, there is again no harm in summing over its polarization states, but now this must be just a sum and <u>not</u> an average.

To summarize, we shall sum $|M|^2$ over the polarizations of all four fermions and insert a factor of 1/2 for each unpolarized particle in the initial state. The results will be valid provided no final state polarizations are measured (except neutrinos) and no initial state particles are polarized (again excepting neutrinos).

For each particle of momentum $\vec{p}$, mass m, a positive energy projection operator $\Lambda_+(\vec{p})$ must be used, and for each antiparticle of momentum $\vec{p}$, mass m, a negative energy projection operator $\Lambda_-(-\vec{p})$ must be used[5].

$$\Lambda_+(\vec{p}) = \frac{\gamma_0\vec{\gamma}\cdot\vec{p} + \gamma_0 m + E}{2E}$$

$$\Lambda_-(-\vec{p}) = \frac{\gamma_0\vec{\gamma}\cdot\vec{p} - \gamma_0 m + E}{2E} ,$$

where E and $\vec{p}$ are the <u>physical</u> values of the energy and momentum of the particle (or antiparticle).

$$\Lambda_+(p)\gamma_0 = \frac{\gamma_0 E - \vec{\gamma}\cdot\vec{p} + m}{2E} ; \quad \Lambda_-(-p)\gamma_0 = \frac{\gamma_0 E - \vec{\gamma}\cdot\vec{p} - m}{2E} .$$

We shall write

$$\Lambda\gamma_0 = \frac{\underset{\sim}{\gamma}\cdot\underset{\sim}{p} \pm m}{2E} \equiv Q ,$$

where the + sign is used for a particle and the - sign for an antiparticle.

$$\underset{\sim}{\gamma}\cdot\underset{\sim}{p} = \sum_{\mu=0}^{3} \gamma_\mu p_\mu g_{\mu\mu} = \gamma_0 E - \vec{\gamma}\cdot\vec{p} .$$

The result of inserting these projection operators and using the closure relation is[5]

$$\sum_{\substack{\text{all}\\ \text{polari-}\\ \text{zations}}} |M|^2 = \frac{G^2}{2} \sum_{\mu,\nu} \mathrm{Tr}(S_{\mu\nu})\mathrm{Tr}(T_{\mu\nu}) g_{\mu\mu} g_{\nu\nu} ,$$

where $S_{\mu\nu} \equiv \gamma_\mu a Q_B \gamma_\nu a Q_A$, $T_{\mu\nu} \equiv \gamma_\mu b Q_D \gamma_\nu b Q_C$, and Tr denotes the trace. Now and hereafter summations over Greek indices are understood to run from zero to three.

Commuting the b which is to the right in $T_{\mu\nu}$ towards the left via the following sequence of steps

$$\gamma_\nu b = (f + gi\gamma_5)\gamma_\nu$$

$$Q_D\gamma_\nu b = \frac{1}{2E_D}\left[(f - gi\gamma_5)\underset{\sim}{\gamma}\cdot\underset{\sim}{p}_D \pm (f + gi\gamma_5)m_D\right]\gamma_\nu$$

$$bQ_D\gamma_\nu b = \frac{1}{2E_D}\left[2c(1 - hi\gamma_5)\underset{\sim}{\gamma}\cdot\underset{\sim}{p}_D \pm 2em_D\right]\gamma_\nu$$

$$\gamma_\mu bQ_D\gamma_\nu b = \frac{1}{2E_D}\left[2c(1 + hi\gamma_5)\gamma_\mu\underset{\sim}{\gamma}\cdot\underset{\sim}{p}_D \pm 2em_D\gamma_\mu\right]\gamma_\nu$$

finally yields

$$T_{\mu\nu} = \frac{1}{2E_DE_C}\left\{\left[c(1 + hi\gamma_5)\gamma_\mu\,\underset{\sim}{\gamma}\cdot\underset{\sim}{p}_D \pm em_D\gamma_\mu\right]\gamma_\nu\left(\underset{\sim}{\gamma}\cdot\underset{\sim}{p}_C \pm m_C\right)\right\} ,$$

where we have put $h \equiv fg/c = 2fg/(f^2 + g^2)$, and $e \equiv (f^2 - g^2)/2$. $S_{\mu\nu}$ has the same form as $T_{\mu\nu}$ with the following replacements: $C \to A$, $D \to B$, $c \to 1$, $h \to 1$, and $e \to 0$.

Most of the formulas needed for evaluating the trace of $T_{\mu\nu}$ are given in the Appendix. In addition there are the results which involve γ_5: γ_5 multiplied by zero, one, two, or three γ-matrices has zero trace; and $\mathrm{Tr}\left(\gamma_5\gamma_\alpha\gamma_\beta\gamma_\mu\gamma_\nu\right) = -4\,\mathcal{E}_{\alpha\beta\mu\nu}$, where $\mathcal{E}$ is a completely antisymmetric tensor. $\mathcal{E}$ vanishes if any two subscripts are equal; $\mathcal{E}_{0123} = +1$; all even permutations of (0123) have $\mathcal{E} = +1$; if the permutation of the subscripts is odd, $\mathcal{E} = -1$.

$$E_C E_D \mathrm{Tr}(T_{\mu\nu}) = \pm 2 m_C m_D e g_{\mu\nu}$$

$$+2c \sum_{\alpha,\beta} p_{D_\alpha} p_{C_\beta} g_{\alpha\alpha} g_{\beta\beta} \left(g_{\mu\alpha} g_{\nu\beta} - g_{\mu\nu} g_{\alpha\beta} + g_{\mu\beta} g_{\nu\alpha} \right)$$

$$-2chi \sum_{\alpha,\beta} p_{D_\alpha} p_{C_\beta} g_{\alpha\alpha} g_{\beta\beta} \mathcal{E}_{\mu\alpha\nu\beta} \quad ,$$

where now the plus sign in the first term is to be used if C and D are both particles or both antiparticles, and the minus sign is to be used if one is a particle and the other an antiparticle. The summations in the second term can be simply performed,

$$E_C E_D \mathrm{Tr}(T_{\mu\nu}) =$$

$$\pm 2 m_C m_D e g_{\mu\nu} + 2c \left(p_{D_\mu} p_{C_\nu} + p_{D_\nu} p_{C_\mu} - g_{\mu\nu} \, \underset{\sim}{p_D} \cdot \underset{\sim}{p_C} \right) - 2chi \sum_{\alpha,\beta} p_{D_\alpha} p_{C_\beta} g_{\alpha\alpha} g_{\beta\beta} \mathcal{E}_{\mu\alpha\nu\beta} \quad .$$

With the same substitutions as before,

$$E_A E_B \mathrm{Tr} \left(S_{\mu\nu} \right) = 2 \left(p_{B_\mu} p_{A_\nu} + p_{B_\nu} p_{A_\mu} - g_{\mu\nu} \, \underset{\sim}{p_B} \cdot \underset{\sim}{p_A} \right) - 2i \sum_{\lambda,\sigma} p_{B_\lambda} p_{A_\sigma} g_{\lambda\lambda} g_{\sigma\sigma} \mathcal{E}_{\mu\lambda\nu\sigma} \quad .$$

Using the definitions of g and $\mathcal{E}$, one gets

$$E_AE_BE_CE_D\sum_{\mu,\nu}\mathrm{Tr}\left(T_{\mu\nu}\right)\mathrm{Tr}\left(S_{\mu\nu}\right)g_{\mu\mu}g_{\nu\nu} = \mp 8m_Cm_D e\,\underset{\sim}{p_A}\cdot\underset{\sim}{p_B}$$

$$+8c\left[\left(\underset{\sim}{p_A}\cdot\underset{\sim}{p_C}\right)\left(\underset{\sim}{p_B}\cdot\underset{\sim}{p_D}\right) + \left(\underset{\sim}{p_A}\cdot\underset{\sim}{p_D}\right)\left(\underset{\sim}{p_B}\cdot\underset{\sim}{p_C}\right)\right]$$

$$-8ch\left[\left(\underset{\sim}{p_A}\cdot\underset{\sim}{p_D}\right)\left(\underset{\sim}{p_B}\cdot\underset{\sim}{p_C}\right) - \left(\underset{\sim}{p_B}\cdot\underset{\sim}{p_D}\right)\left(\underset{\sim}{p_A}\cdot\underset{\sim}{p_C}\right)\right] ,$$

where we have used the fact that $\sum_{\mu,\nu} g_{\mu\mu}g_{\nu\nu}\varepsilon_{\mu\alpha\nu\beta}\varepsilon_{\mu\lambda\nu\sigma} = 2\left(g_{\alpha\sigma}g_{\beta\lambda} - g_{\alpha\lambda}g_{\beta\sigma}\right)$ to simplify the last term.

The result of this section can be summarized in the following formula, where c and h have been reexpressed in terms of the coupling constants f and g.

$$\sum_{\substack{\text{all}\\ \text{polari-}\\ \text{zations}}} |M|^2 = \frac{8G^2}{E_AE_BE_CE_D}\left[\frac{(f+g)^2}{4}\left(\underset{\sim}{p_A}\cdot\underset{\sim}{p_C}\right)\left(\underset{\sim}{p_B}\cdot\underset{\sim}{p_D}\right)\right.$$

$$\left. + \frac{(f-g)^2}{4}\left(\underset{\sim}{p_A}\cdot\underset{\sim}{p_D}\right)\left(\underset{\sim}{p_B}\cdot\underset{\sim}{p_C}\right) \mp \frac{(f^2-g^2)}{4} m_Cm_D\underset{\sim}{p_A}\cdot\underset{\sim}{p_B}\right] .$$

The minus sign in the last term is used if C and D are both particles or both antiparticles; otherwise, the plus sign is used. Note that the expression in brackets is a Lorentz invariant quantity.

IV. Kinematics

Given two particles in the final state, labeled 1 and 2, with total momentum-energy $\vec{p}$, E,

$$\vec{p} = \vec{p}_1 + \vec{p}_2 \qquad E = E_1 + E_2 \quad ,$$

the energy-angle relation for particle 1 can be found by eliminating $\vec{p}_2$ and E_2.

$$(\vec{p} - \vec{p}_1)^2 = p_2^2 = E_2^2 - m_2^2 = (E - E_1)^2 - m_2^2 \quad ,$$

which can be written as

$$EE_1 - pp_1 \cos\theta = \Delta \quad ,$$

where $2\Delta \equiv \mathcal{E}^2 + m_1^2 - m_2^2$, $\mathcal{E}^2 \equiv E^2 - \vec{p}^2 = m_1^2 + m_2^2 + 2\underset{\sim}{p_1}\cdot\underset{\sim}{p_2}$, with θ the angle between $\vec{p}_1$ and $\vec{p}$ (the total momentum). $\mathcal{E}^2$ is the invariant square of the total energy-momentum four vector. Squaring and rearranging yields a quadratic equation for E_1

$$(E^2 - p^2 \cos^2\theta)\, E_1^2 - 2\Delta E E_1 + \Delta^2 + m_1^2\, p^2 \cos^2\theta = 0 \quad ,$$

which can be solved to give E_1 as a function of θ,

$$E_1 = \frac{E\left(\mathcal{E}^2 + m_1^2 - m_2^2\right) \pm p|\cos\theta| \sqrt{\left[\mathcal{E}^2 - (m_1^2 + m_2^2)\right]^2 - 4m_1^2(m_2^2 + p^2 \sin^2\theta)}}{2(E^2 - p^2 \cos^2\theta)} \quad .$$

The maximum and minimum values of E_1 can be found from this by putting $\sin\theta = 0$.

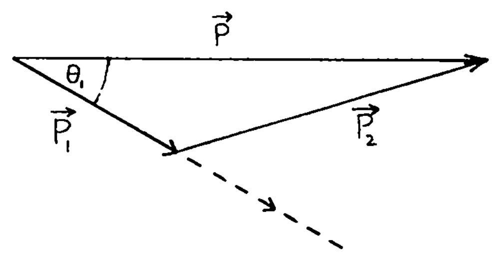
If the total energy and momentum is due to two particles in the initial state, labeled 3 and 4, then $\mathcal{E}^2 = m_3^2 + m_4^2 + 2\underset{\sim}{p_3}\cdot\underset{\sim}{p_4}$.

In the laboratory system, with particle 4 originally at rest,

$\vec{p} = \vec{p}_3$, $E = E_3 + m_4$, $\mathcal{E}^2 = m_3^2 + m_4^2 + 2E_3m_4$. In this case $\underset{\sim}{p_3} \cdot \underset{\sim}{p_1} = E_3E_1 - \vec{p}_3 \cdot \vec{p}_1 = \Delta - m_4E_1$.

V. Density of States

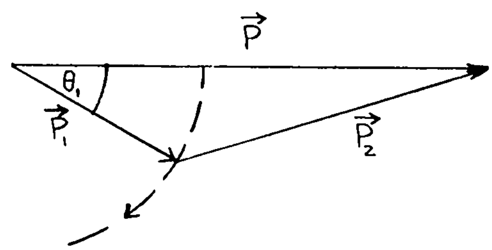
Given a system of total momentum $\vec{p}$ and total energy E which disintegrates into two particles ($\vec{p} = \vec{p}_1 + \vec{p}_2$; $E = E_1 + E_2$), one can get expressions for the number of states per unit total energy in different forms depending upon what physical variables are chosen. Williams[7] gives the derivation for the case in which the density of states ρ is expressed in terms of the element of solid angle of one of the particles.



That derivation is poor in that it fails to take account of the fact that (for fixed values of p and θ_1) E may decrease as p_1 increases. However, the result of that calculation is acceptable provided absolute values are taken:

$$\rho = \frac{1}{(2\pi)^3} \; E_1E_2 \; \frac{p_1^3 d\Omega_1}{\left| Ep_1^2 - E_1 \vec{p} \cdot \vec{p}_1 \right|} \quad .$$

If one chooses to express ρ in terms of the element of energy of one of the particles, then this ambiguity is not present since for fixed p and E_1, E is a monotonically increasing function of θ_1 (just because p_2 is a monotonically increasing function of θ_1).



The idea behind the Williams' derivation, as applied to the variable E_1, is to find the total number of states with energy less than or equal to E, with fixed E_1 and $\vec{p}$ (and φ_1), and then differentiate this expression with respect to E to find the number of states per unit total energy.

$$\rho = \frac{d}{dE}\left[\frac{d\varphi_1}{(2\pi)^3}\int p_1^2 \, dp_1 \; d(-\cos\theta_1)\right] \quad ,$$

where the range of integration is from $\theta_1 = 0$ to that value of θ_1 which makes $(E_1 + E_2)$ equal to E, i.e., the physical value of θ_1 for the variables p and E.

$$\rho = \frac{1}{(2\pi)^3} p_1 E_1 dE_1 d\varphi_1 \left[\frac{d(-\cos\theta_1)}{dE}\right]_{\text{constant } E_1}$$

$$E = E_1 + E_2 = E_1 + \sqrt{m_2^2 + (\vec{p}-\vec{p}_1)^2}$$

$$= E_1 + \sqrt{m_2^2 + p^2 + p_1^2 - 2pp_1 \cos\theta_1}$$

$$-\frac{dE}{d\cos\theta_1} = \frac{pp_1}{E_2} \quad .$$

$$\rho = \frac{1}{(2\pi)^3} \frac{E_1 E_2}{p} dE_1 d\varphi_1 \quad .$$

Of course this result can be obtained directly from the formula in terms of solid angle by using the relation (which can be simply derived from Section IV)

$$pp_1^3 \frac{d\cos\theta_1}{dE_1} = Ep_1^2 - \vec{p}\cdot\vec{p}_1 E_1 \quad .$$

VI. Cross Sections

The cross section for a reaction induced by neutrinos or antineutrinos on an unpolarized target, as explained in Section III, is

$$\sigma = 2\pi\cdot\frac{1}{2} \sum_{\substack{\text{all} \\ \text{polari-} \\ \text{zations}}} |M|^2 \cdot \rho \quad ,$$

where we have still to insert the necessary factors of $\hbar$ and c. To do this, observe that $|M|^2\rho$ has the dimension of $[\text{Energy}]^4 \cdot [\text{Length}]^6$ with two powers of energy coming from G^2 and two powers coming from particle energies. Multiplying by $(\hbar c)^{-4}$ will give the proper dimensions for the cross section. Defining[1]

$$\sigma_o \equiv \frac{2}{\pi} G^2 \frac{(mc^2)^2}{(\hbar c)^4} = \frac{2}{\pi} (1.01 \times 10^{-5})^2 \left[\frac{(\hbar c)(mc^2)}{(M_p c^2)^2}\right]^2 = 8.5_3 \times 10^{-45} \text{ cm}^2 \quad ,$$

where m is the electron mass, yields the result

$$\frac{d\sigma}{dE_1} = \sigma_o \frac{1}{E_A E_B E_C E_D} \frac{E_1 E_2}{p} \left[\frac{(f+g)^2}{4} \left(\underset{\sim}{p}_A \cdot \underset{\sim}{p}_C\right) \left(\underset{\sim}{p}_B \cdot \underset{\sim}{p}_D\right) + \frac{(f-g)^2}{4} \left(\underset{\sim}{p}_A \cdot \underset{\sim}{p}_D\right) \left(\underset{\sim}{p}_B \cdot \underset{\sim}{p}_C\right) \mp \frac{(f^2-g^2)}{4} m_C m_D \, \underset{\sim}{p}_A \cdot \underset{\sim}{p}_B \right] \quad ,$$

where all masses are expressed in units of m, all momenta in units of mc, and all energies in units of mc^2. E_1 and E_2 are the two final particle energies, and we have chosen to express the cross section per unit of one of the final particle energies rather than per unit solid angle. The minus sign in the last term is used if C and D are both particles or both antiparticles; the plus sign is used otherwise.

VII. Examples

A. Antineutrino-Electron Scattering

If the four-fermion interaction arises from the interaction of a current with itself[1], $\sum_\mu g_{\mu\mu} J_\mu J_\mu^+$, where one of the terms in J_μ is due to an electron-neutrino combination, $\bar{\Psi}_\nu \gamma_\mu (1 - i\gamma_5)\psi_e$, then antineutrino-electron scattering and neutrino-electron scattering will both occur. If there is a charged intermediate boson they must occur. (We are now talking about the neutrino which is associated with the electron ν_e, not the neutrino which is associated with the μ-meson.) We first consider the reaction

$$\tilde{\nu}_\alpha + e_\alpha \rightarrow \tilde{\nu}_\beta + e_\beta \quad ,$$

where the subscripts are used for identification purposes. The first step is to properly identify the four particles taking part in the reaction with the four labels A, B, C, and D: $A = \tilde{\nu}_\alpha$, $B = e_\alpha$, $C = e_\beta$, and $D = \tilde{\nu}_\beta$. (One could, in this example, simultaneously interchange A with C and B with D.) This means that the particles are paired as shown

$$\underbrace{\tilde{\nu}_\alpha + e_\alpha} \rightarrow \underbrace{\tilde{\nu}_\beta + e_\beta} \quad .$$

In this reaction the form factors f and g should both be put equal to unity. From the general result of Section VI,

$$d\sigma = \sigma_o \frac{dE_{e_\beta}}{E_{\nu_\alpha} E_{e_\alpha} p} \left(\underset{\sim}{p}_{\nu_\alpha} \cdot \underset{\sim}{p}_{e_\beta}\right)\left(\underset{\sim}{p}_{e_\alpha} \cdot \underset{\sim}{p}_{\nu_\beta}\right) \quad ,$$

where all masses, energies, and momenta are in units of m, mc^2 and mc, respectively.

To facilitate the specialization of this result to the laboratory system, make use of the labeling system from Section IV: $1 = e_\beta$, $2 = \nu_\beta$, $3 = \nu_\alpha$, and $4 = e_\alpha$. We have already made use of the fact that 1 and 2 are the two final particles; particle 3 is the one which is incident in the laboratory, and particle 4 is initially at rest. From Section IV

$$p = p_3 = p_{\nu_\alpha}, \; E_{e_\alpha} = E_4 = m$$

$$\mathcal{E}^2 = m^2 + 2mE_{\nu_\alpha}, \; \Delta = m\left(m + E_{\nu_\alpha}\right)$$

$$\underset{\sim}{p}_{\nu_\alpha}\cdot\underset{\sim}{p}_{e_\beta} = \underset{\sim}{p}_3\cdot\underset{\sim}{p}_1 = m\left(m + E_{\nu_\alpha} - E_{e_\beta}\right)$$

$$\underset{\sim}{p}_{e_\alpha}\cdot\underset{\sim}{p}_{\nu_\beta} = \underset{\sim}{p}_4\cdot\underset{\sim}{p}_2 = mE_{\nu_\beta} = m\left(m + E_{\nu_\alpha} - E_{e_\beta}\right) \quad ,$$

where conservation of energy gives the last form of the last equation. Introducing $T = E_{e_\beta} - m$, the kinetic energy of the recoil electron, and putting together the results just obtained gives

$$d\sigma_{LAB} = \left(\frac{\sigma_o}{mc^2}\right)\left(1 - \frac{T}{E_\nu}\right)^2 dT \quad ,$$

where $E_\nu \equiv E_{\nu_\alpha}$, the incident antineutrino energy, and $(\sigma_o/mc^2) = 1.67 \times 10^{-44}\ cm^2 - MeV^{-1}$.

The next step is to determine the maximum and minimum possible values of T. From Section IV these are found to be

$$T_{min} = 0, \; T_{max} = \frac{2E_\nu^2}{mc^2 + 2E_\nu} \quad ,$$

with the minimum (maximum) T coming from electrons which go backward (forward) in the center of mass system. $\left(\frac{d\sigma}{dT}\right)_{LAB}$ is plotted vs T on Figure I. The cutoff at T_{max} is sharp. The total cross section is found by integrating over all recoil energies,

$$\sigma_T = \sigma_o \, \frac{1}{3} \, \frac{E_\nu}{mc^2} \left[1 - \left(\frac{mc^2}{mc^2 + 2E_\nu} \right)^3 \right] \quad .$$

Note that this result agrees with that obtained in reference 4, but is exactly twice as large as the value quoted in footnote 17 of reference 1. The total cross section is plotted on Figure II. The departure from linearity at small energies is hardly visible on this scale.

One point which still has to be discussed is the possible existence of neutral symmetric currents, terms such as $\overline{\psi}_\nu \gamma_\mu (1 - i\gamma_5) \psi_\nu$ and $\overline{\psi}_e \gamma_\mu (1 - i\gamma_5) \psi_e$. These terms will certainly be present if there is a neutral intermediate boson. If they are present, then the scattering which we have been studying can also proceed via this type of coupling

$$\tilde{\nu} + e \to \tilde{\nu} + e \quad .$$

It is shown in footnote 7 of reference 1 that the matrix element for this coupling has exactly the same magnitude as the matrix element for the other coupling scheme (the one we have used), but that the relative sign is not certain. This sign is positive if the neutrino field operator and the electron field operator anticommute; it is negative if they commute. Since the matrix elements must be added together (if both couplings are present) there is the possibility of obtaining, in the cross section, a factor of zero or a factor of four. This assumes that the coupling constant is the same for both types of currents, and there is no a priori reason for this to be the case.

An example of a reaction which can proceed only if neutral currents exist is $\tilde{\nu}_\mu + e \to \tilde{\nu}_\mu + e$. The observation of this reaction with the cross section computed above would be very strong evidence for neutral currents with the same coupling constant as that for the charged currents. In reference 8 it is demonstrated that $\tilde{\nu}_\mu$ - e scattering can take place even if there are only charged intermediate bosons, but that the process is of higher order and is expected to have a cross section very much smaller than if there were neutral currents.

B. Neutrino-Electron Scattering

$$\nu_\alpha + e_\alpha \to \nu_\beta + e_\beta \quad .$$

The coupling is as shown; i.e., make the identifications $A = e_\beta = 1$, $B = \nu_\alpha = 3$, $C = \nu_\beta = 2$, and $D = e_\alpha = 4$. Again $f = g = 1$.

$$\underset{\sim}{p}_A \cdot \underset{\sim}{p}_C = \underset{\sim}{p}_{e_\beta} \cdot \underset{\sim}{p}_{\nu_\beta} , \quad \underset{\sim}{p}_B \cdot \underset{\sim}{p}_D = \underset{\sim}{p}_{\nu_\alpha} \cdot \underset{\sim}{p}_{e_\alpha} ,$$

and from Section IV both these products are equal to mE_{ν_α} (LAB). From the general result of Section VI

$$d\sigma_{LAB} = \left(\frac{\sigma_o}{mc^2}\right) dT ,$$

where again T is the kinetic energy of the recoil electron. The kinematics is exactly the same as in Example A. The spectrum of recoils is flat from $T = 0$ to $T = 2E_\nu^2 \big/ \left(mc^2 + 2E_\nu\right)$, and is plotted on Figure I. The total cross section is therefore

$$\sigma_T = \sigma_o \frac{2E_\nu^2}{mc^2(mc^2+2E_\nu)} ,$$

which is plotted on Figure II. This result agrees with reference 4, but is again twice as large as the value quoted in reference 1.

Just as in Example A, the existence of neutral currents would alter these results.

C. Antineutrino Absorption on Protons

$$\tilde{\nu} + p \rightarrow n + e^+ .$$

The identification required here is $A = \tilde{\nu}$, $B = \tilde{e}$, $C = n$, and $D = p$. The coupling constants which fit nuclear beta-decay are $f = 1$, $g = 1.2$,

$$\frac{(f+g)^2}{4} = 1.2_0 \; ; \quad \frac{(f-g)^2}{4} = 0.01 \; ; \quad \frac{-(f^2-g^2)}{4} = 0.1_1$$

At energies much smaller than the nucleon rest mass, the recoil energy of the neutron is negligible, and $E_e \cong E_\nu - (M_n - M_p)$ (all energies are in the laboratory system). Since the center of mass system is identical with the laboratory system, it is necessary to use the solid angle form of the density of states

$$\rho = \frac{1}{(2\pi)^3} \; \frac{E_e E_n p_e^3 d\Omega_e}{E p_e^2 - E_e \vec{p}_e \cdot \vec{p}_\nu} \quad .$$

The total energy E is approximately the nucleon mass as is E_n, so the second term in the denominator is negligible (above $E_\nu = 2$ MeV), and

$$\rho \cong \frac{d\Omega}{(2\pi)^3} \; E_e p_e$$

with

$$p_e = \sqrt{E_e^2 - m^2} \quad .$$

$$\underset{\sim}{p_A} \cdot \underset{\sim}{p_C} = \underset{\sim}{p_\nu} \cdot \underset{\sim}{p_n} = E_\nu M_n$$

$$\underset{\sim}{p_B} \cdot \underset{\sim}{p_D} = \underset{\sim}{p_e} \cdot \underset{\sim}{p_p} = E_e M_p$$

$$\underset{\sim}{p_A} \cdot \underset{\sim}{p_B} = \underset{\sim}{p_\nu} \cdot \underset{\sim}{p_e} = E_\nu E_e - p_\nu p_e \cos\theta$$

$$\underset{\sim}{p_A} \cdot \underset{\sim}{p_D} = \underset{\sim}{p_\nu} \cdot \underset{\sim}{p_p} = E_\nu M_p$$

$$\underset{\sim}{p_B} \cdot \underset{\sim}{p_C} = \underset{\sim}{p_e} \cdot \underset{\sim}{p_n} = E_e M_n \quad ,$$

where θ is the angle between the positron's direction and that of the incoming antineutrino. Using the formula for $|M|^2$ at the end of Section III gives

$$d\sigma_{LAB} = \frac{\sigma_o d\Omega}{2\pi}\left[1.2_0 E_e p_e + 0.1_1 p_e \left(E_e - p_e \cos\theta\right) + 0.01\, E_e p_e\right] \quad ,$$

where again the energies and momenta are dimensionless, being referred to mc^2 and mc.

$$d\sigma_{LAB} = \frac{1.3_2\sigma_o d\Omega}{2\pi} \left(\frac{E_\nu}{mc^2} - 2.53\right) \sqrt{\left(\frac{E_\nu}{mc^2} - 2.53\right)^2 - 1} \times$$

$$\times \left[1 - 0.08 \frac{\sqrt{\left(\frac{E_\nu}{mc^2} - 2.53\right)^2 - 1}}{\left(\frac{E_\nu}{mc^2} - 2.53\right)} \cos\theta\right] \quad .$$

Note that there is already an anisotropy of 7% with 2.5 MeV antineutrinos. The cos θ term does not contribute to the total cross section, which is simply

$$\sigma_T = 2.6_4\sigma_o \left(\frac{E_\nu}{mc^2} - 2.53\right) \sqrt{\left(\frac{E_\nu}{mc^2} - 2.53\right)^2 - 1} \quad .$$

The threshold is at $E_\nu \cong (M_n - M_p) + m = 1.805$ MeV. The total cross section is plotted on Figure III.

REFERENCES

1. Feynman and Gell-Mann, Phys. Rev., 109, 193 (1958).
2. Gell-Mann, Phys. Rev., 111, 362 (1958); Lee, Mo, and Wu, Phys. Rev. Letters, 10, 253 (1963).
3. Goldberger and Treiman, Phys. Rev., 111, 354 (1958).
4. Azimov and Shekhter, Soviet Phys. JETP, 14, 424 (1962).
5. See Appendix for details about the closure relation and energy projection operators.
6. See "An Introduction to Relativistic Quantum Field Theory" by S. S. Schweber, 1961, pp. 498-500, for projection operators which single out a given polarization.
7. See, for example, "An Introduction to Elementary Particles" by W. S. C. Williams, 1961, Appendix B.
8. Feinberg and Pais, Phys. Rev., 131, 2724 (1963).

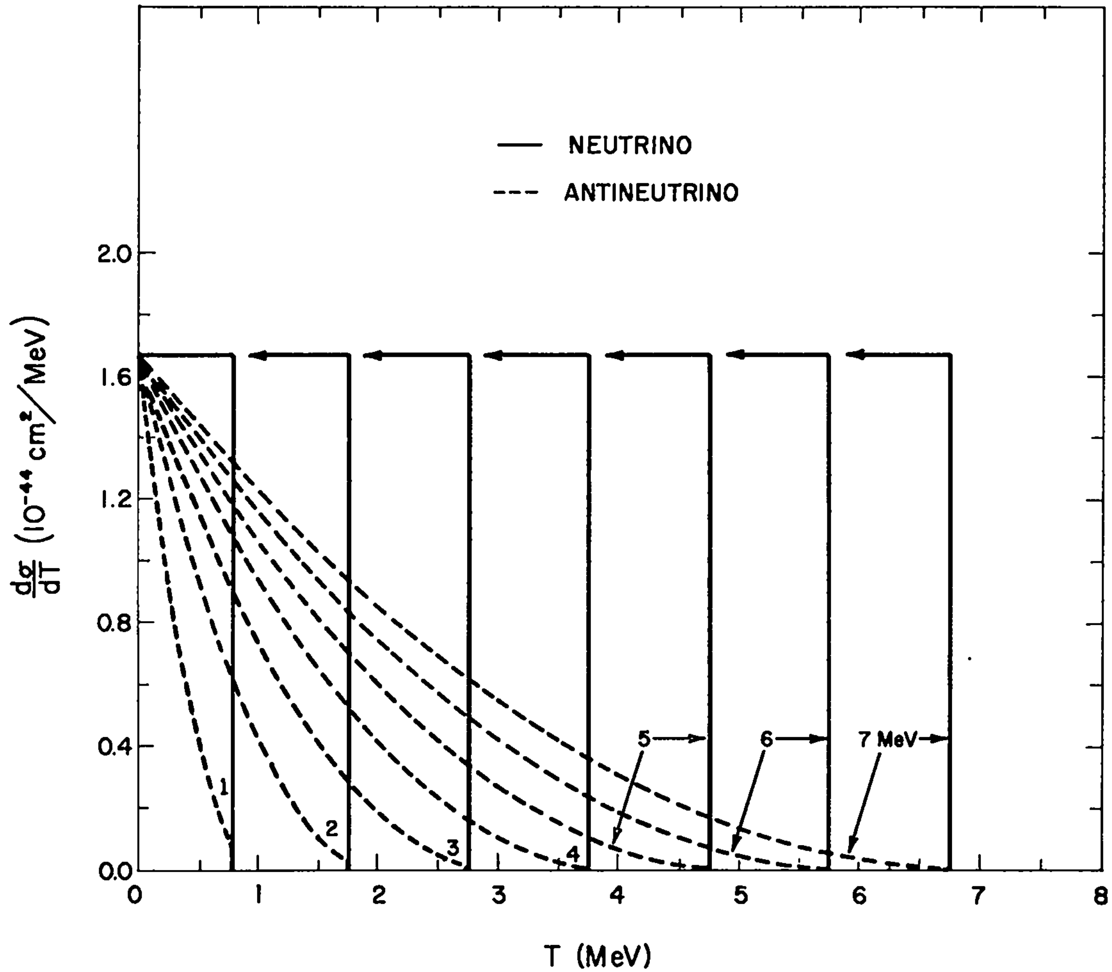


Figure I. Laboratory differential cross section per unit electron recoil kinetic energy T, vs T, for elastic neutrino and antineutrino scattering on electrons. The curves are labeled by the incident neutrino or antineutrino laboratory energy in MeV. Note that the electron recoil spectrum is flat from $T = 0$ to T_{max} for the case of incident neutrinos. The calculation assumes that there are no neutral symmetric currents.

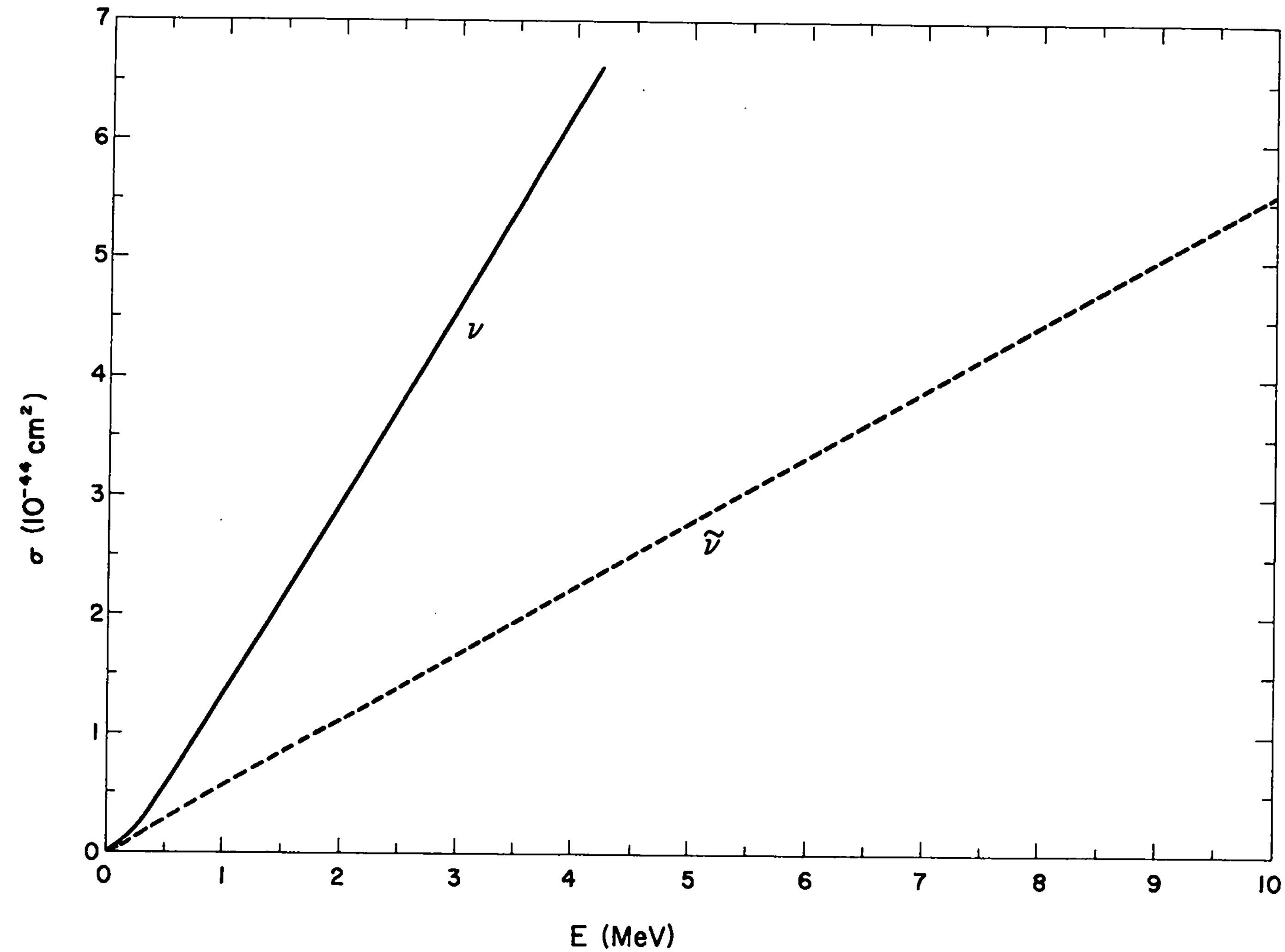


Figure II. Total cross section for elastic neutrino and antineutrino scattering on electrons vs the incident neutrino or antineutrino laboratory energy. The calculation assumes that there are no neutral symmetric currents.

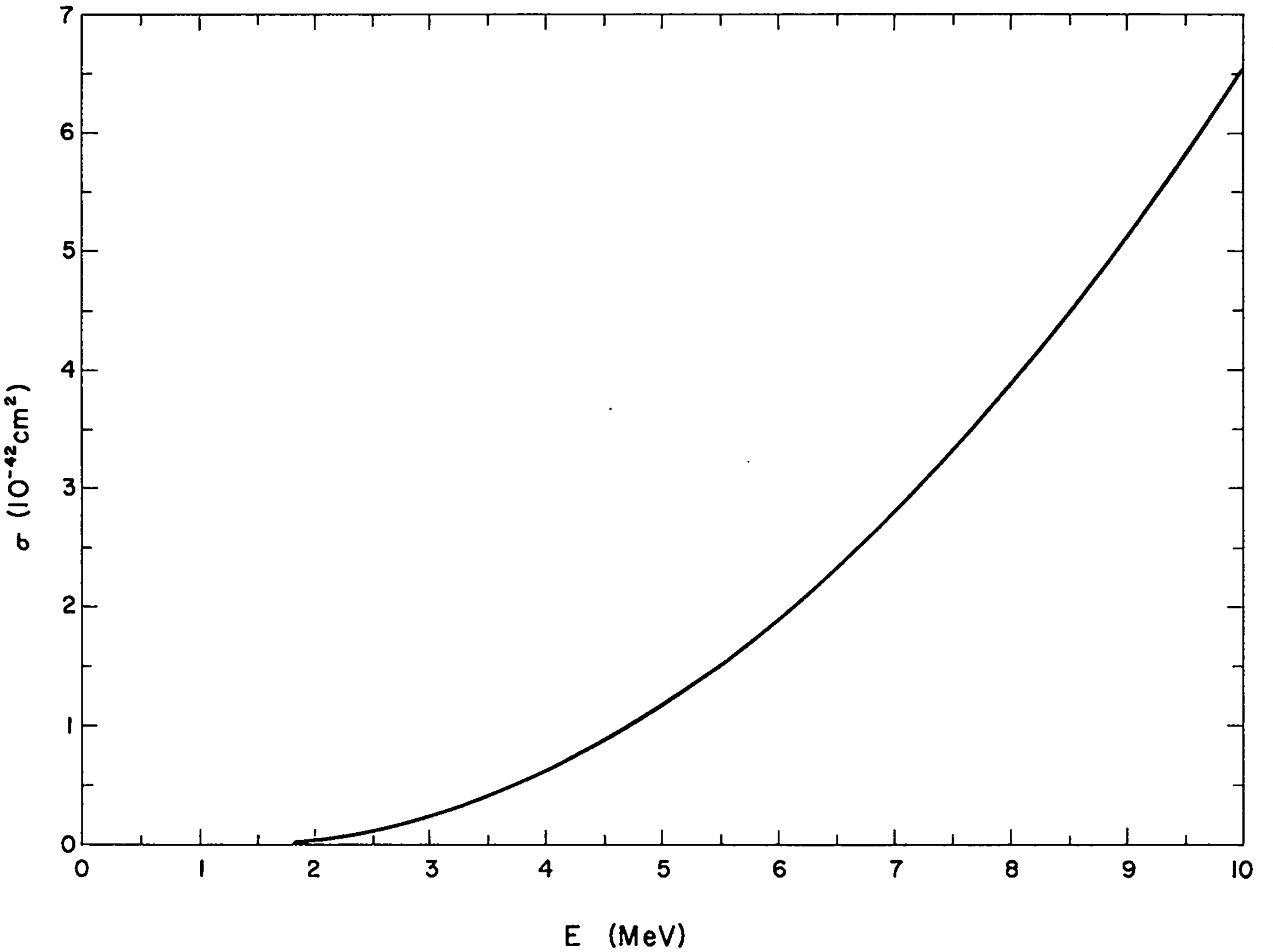


Figure III. Total cross section for $\bar{\nu} + p \rightarrow n + e^{+}$ vs the incident antineutrino laboratory energy. The anisotropy (which is less than 8%) is given in the text.

APPENDIX

Lectures on Beta-Decay Theory

Los Alamos Scientific Laboratory, Los Alamos, New Mexico

Fall 1957

II The Energy-Angle Distribution in Old-Fashioned Beta-Decay

L. Heller

In the following, the Coulomb effect on the electron will be omitted. The main modification due to the Coulomb field is to put a factor $F(Z,W)$ into the spectrum. This factor cuts down on the number of positrons, and increases the number of electrons, especially at low energies. $F(Z,W)$ can be found on p. 280 of the review article by Rose.*

Using the notation of K. Ford's lectures,** the transition probability per unit time is written:

$$T = \frac{2\pi}{\hbar} |M|^2 (2\pi)^{-6} \sqrt{W^2-1}\; W(W_o-W)^2\, dW d\Omega_e d\Omega_\nu \quad ,$$

where

$$M = \sum_n C_n \left\langle \psi_f \middle| \sum_j \beta_j O_{nj} \tau_{-j} \varphi_e^*(\vec{x}_j) \beta O_n \varphi_{\bar{\nu}}(\vec{x}_j) \middle| \psi_o \right\rangle \quad .$$

ψ_o is the initial state of the nucleus, ψ_f the final state, and j goes over all the neutrons (for negative electron decay) in the nucleus. In the absence of the Coulomb field we can write

$$\varphi_e(\vec{x}) = \varphi_e(\vec{p})\, e^{i\vec{p}\cdot\vec{x}}$$

$$\varphi_{\bar{\nu}}(\vec{x}) = \varphi_{\bar{\nu}}(-\vec{q})\, e^{-i\vec{q}\cdot\vec{x}} \quad ,$$

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*M. E. Rose 1955, Beta and Gamma-Ray Spectroscopy, ed. K. Siegbahn, Chap. IX.

**K. Ford 1957, Lectures on Beta-Decay Theory, Los Alamos Scientific Laboratory, I. "Old Fashioned Beta-Decay Theory".

since the negative energy state with momentum $-\vec{q}$ corresponds to the emitted neutrino having momentum $+\vec{q}$. The functions φ on the right-hand sides of these equations are spinors which are independent of position.

Now the light particle functions can be separated from the nuclear functions.

$$M = \sum_n C_n \left\langle \psi_f \right| \sum_j e^{-i(\vec{p}+\vec{q})\cdot\vec{x}_j} \beta_j O_{nj} \tau_{-j} \left| \psi_i \right\rangle \left[\varphi_e^*(\vec{p}) \beta O_n \varphi_{\bar{\nu}}(-\vec{q}) \right] .$$

We will use the notation $\int f(x)\beta O_n$ to represent the nuclear matrix element, where $f(x)$ is that part of $e^{-i(\vec{p}+\vec{q})\cdot\vec{x}}$ which is under consideration at the time.

$$M = \sum_n C_n \left[\int f(x)\beta O_n \right] \left[\varphi_e^*(\vec{p}) \beta O_n \varphi_{\bar{\nu}}(-\vec{q}) \right] .$$

These matrix elements are to be taken between particular spin states of the particles involved.

$$|M|^2 = \sum_n \sum_m C_n^* C_m \left[\int f(x)\beta O_n \right]^* \left[\int f(x)\beta O_m \right] \left[\varphi_e^*(\vec{p}) \beta O_m \varphi_{\bar{\nu}}(-\vec{q}) \right] \left[\varphi_e^*(\vec{p}) \beta O_n \varphi_{\bar{\nu}}(-\vec{q}) \right]^* .$$

φ represents a one-column, four-row spinor, and φ^* represents that one-row, four-column spinor which has as entries the complex conjugates of the entries of φ.

By well-known matrix rules,

$$\left(\varphi_e^* \beta O_n \varphi_{\bar{\nu}}\right)^* = \varphi_{\bar{\nu}}^* O_n^+ \beta \varphi_e \quad ,$$

where O_n^+ is the Hermitean conjugate of O_n. β, in our notation is Hermitean. Finally

$$|M|^2 = \sum_n \sum_m C_n^* C_m \left[\int f(x) \beta O_n\right]^* \left[\int f(x) \beta O_m\right] \left[\varphi_e^*(\vec{p}) \beta O_m \varphi_{\bar{\nu}}(-\vec{q}) \varphi_{\bar{\nu}}^*(-\vec{q}) O_n^+ \beta \varphi_e(\vec{p})\right] \quad .$$

Introduce the notation $(\vec{p}, -\vec{q})_{mn}$ for the quantity in parentheses.

To compute $|M|^2$, these operators have to be taken two at a time, (e.g., $O_m = \beta$, $O_n = \beta$; or $O_m = \beta$, $O_n = \sigma_x$, etc.), put in whatever spinors one is interested in, carry out the matrix products, and sum over all values of m and n. This last step involves the nuclear matrix elements. A tremendous simplification results if one is not interested in the spin of the light particles, i.e., if one sums $|M|^2$ over all possible spins of electron and neutrino. We can do this for each pair (m,n) and then sum over m and n. To see how this comes about, we can put in the explicit spinors and carry out the operations. The work can be greatly simplified by using projection operators. We will make a slight digression onto this topic.

PROJECTION OPERATORS

The reason projection operators are useful can be seen from the closure theorem for complete orthonormal functions. This theorem says that

$$\sum_{j=1}^{4} u^j(\vec{p})u^{j*}(\vec{p}) = I \qquad \left[\sum_{j=1}^{4} u_\alpha^j(\vec{p})u_\beta^{j*}(\vec{p}) = \delta_{\alpha\beta}\right] \quad ,$$

where $u^j(\vec{p})$ is one of the four independent normalized, orthogonal spinors of momentum $\vec{p}$ (two have positive energy, and two have negative energy). The sum goes over the four states of momentum $\vec{p}$. Note that $u^j(\vec{p})$, which is a column vector comes first, and $u^{j*}(\vec{p})$, a row vector, follows. The result of the matrix multiplication for given index j, is a 4×4 matrix. The theorem says that the sum of the four matrices ($j = 1,\ldots 4$) is the identity. The proof follows exactly along the lines given by Schiff*, and is repeated here. An arbitrary spinor of momentum $\vec{p}$ can be written as a linear combination of the four basic spinors $u^j(\vec{p})$

$$\psi(\vec{p}) = \sum_{j=1}^{4} A_j(\vec{p})u^j(\vec{p}) \quad .$$

Multiply both sides on the left by $u^{i*}(\vec{p})$

$$u^{i*}(\vec{p})\psi(\vec{p}) = \sum_{j=1}^{4} A_j(\vec{p})u^{i*}(\vec{p})u^j(\vec{p}) \quad .$$

From the orthonormality of the u's, we have

$$u^{i*}(\vec{p})u^j(\vec{p}) = \delta_{ij} \quad ,$$

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*L. I. Schiff 1949, Quantum Mechanics, First Edition, p. 46.

and therefore

$$u^{i*}(\vec{p})\psi(\vec{p}) = A_i(\vec{p}) \quad .$$

Put this back into the expression for $\psi(\vec{p})$:

$$\psi(\vec{p}) = \sum_{j=1}^{4} u^j(\vec{p})u^{j*}(\vec{p})\psi(\vec{p}) \quad .$$

If this is to be true for arbitrary $\psi(\vec{p})$, we must have

$$\sum_{j=1}^{4} u^j(p)u^{j*}(p) = I, \text{ or} \sum_{j=1}^{4} u^j(\vec{p})_\alpha u^{j*}(p)_\beta = \delta_{\alpha\beta} \quad .$$

Now we have in the middle of the light particle portion of $|M|^2$ the combination $\omega_{\bar{\nu}}(-\vec{q})\omega^*_{\bar{\nu}}(-\vec{q})$, and we want to sum over the final states of the neutrino. If this sum had been over all four of the neutrino spin states having momentum $-\vec{q}$, we could use the theorem just proven and get the identity. However, not all four states are available to the neutrino; it must be in a negative energy state (the way we have set up the Hamiltonian). Similarly, the electron must be in a positive energy state. We want to carry out the following sum

$$\sum_{\substack{\text{Pos En} \\ \text{for } e\ell}} \sum_{\substack{\text{Neg En} \\ \text{for } \bar{\nu}}} (\vec{p},-\vec{q})_{mn} \quad ,$$

which involves

$$\sum_{\substack{\text{Neg En}\\ \text{for } \bar{\nu}}} \varphi_{\bar{\nu}}(-\vec{q})\varphi^*_{\bar{\nu}}(-\vec{q}) \quad .$$

Let us look for operators $\Lambda_+(\vec{p})$and $\Lambda_-(\vec{p})$, with the following properties:

$$\begin{cases} \Lambda_+(\vec{p})\varphi_+(\vec{p}) = \varphi_+(\vec{p}) \\ \Lambda_+(\vec{p})\varphi_-(\vec{p}) = 0 \end{cases}$$

$$\begin{cases} \Lambda_-(\vec{p})\varphi_-(\vec{p}) = \varphi_-(\vec{p}) \\ \Lambda_-(\vec{p})\varphi_+(\vec{p}) = 0 \end{cases} \quad ,$$

where $\varphi_+(\vec{p})$ and $\varphi_-(\vec{p})$ are positive and negative energy states, respectively, of momentum $\vec{p}$.

$\Lambda_+(\vec{p})$ operating on a positive energy state gives it back, and it gives zero on a negative energy state (all of the same momentum). $\Lambda_-(\vec{p})$ has the reverse properties. It is easily seen, using the Dirac equation, that

$$\Lambda_+(\vec{p}) = \frac{\beta\vec{\gamma}\cdot\vec{p}+\beta m+E_+}{2E_+} = \frac{H + E_+}{2E_+}$$

and

$$\Lambda_-(\vec{p}) = \frac{\beta\vec{\gamma}\cdot\vec{p}+\beta m-E_+}{-2E_+} = \frac{H - E_+}{-2E_+}$$

have the desired properties, where E_+ is the absolute value of the energy. (H is the Dirac Hamiltonian: $H = \beta\vec{\gamma}\cdot p+\beta m$)

We can rewrite our sum $\sum\limits_{e\ell_+} \sum\limits_{\bar{\nu}_-} (\vec{p},-\vec{q})_{mn}$ as

$$\sum_{\substack{\text{Pos En}\\ \text{for } e\ell}} \sum_{\substack{\text{Pos \&}\\ \text{Neg En}\\ \text{for } \bar{\nu}}} \left[\varphi_e^*(\vec{p})\beta O_m\Lambda_-(-\vec{q})\varphi_{\bar{\nu}}(-\vec{q})\varphi_{\bar{\nu}}^*(-\vec{q})O_n^+\beta\varphi_e(\vec{p})\right] ,$$

since Λ_- gives zero on the positive energy states and just gives back the negative energy states.

Now the closure theorem can be used on the neutrino, since the sum goes over all four spin states. We get

$$\sum_{e\ell_+} \sum_{\bar{\nu}_-} (\vec{p},-\vec{q})_{mn} = \sum_{e\ell_+} \left[\varphi_e^*(\vec{p})\beta O_m\Lambda_-(-\vec{q})O_n^+\beta\varphi_e(\vec{p})\right] .$$

We can do the same trick on the electron by introducing a positive energy projection operator for it.

$$\sum_{e\ell_+} \sum_{\bar{\nu}_-} (\vec{p},-\vec{q})_{mn} = \sum_{\substack{e\ell_+\\ \text{and}\\ e\ell_-}} \left[\varphi_e^*(\vec{p})\beta O_m\Lambda_-(-\vec{q})O_n^+\beta\Lambda_+(\vec{p})\varphi_e(\vec{p})\right] .$$

By grouping the matrices, this really means

$$\sum_{\alpha=1}^{4}\sum_{\beta=1}^{4}\sum_{\substack{e\ell_+ \\ \text{and} \\ e\ell_-}}\left\{\varphi_e^*(\vec{p})_\alpha\left[\beta O_m\Lambda_-(-\vec{q})O_n^+\beta\Lambda_+(\vec{p})\right]_{\alpha\beta}\varphi_e(\vec{p})_\beta\right\} \quad ,$$

which can be rewritten as

$$\sum\sum\sum\left\{\left[\beta O_m\Lambda_-(-\vec{q})O_n^+\beta\Lambda_+(\vec{p})\right]_{\alpha\beta}\varphi_e(\vec{p})_\beta\varphi_e^*(\vec{p})_\alpha\right\} \quad .$$

Again, using the closure relation gives

$$\sum_\alpha\sum_\beta\left\{\left[\beta O_m\Lambda_-(-\vec{q})O_n^+\beta\Lambda_+(\vec{p})\right]_{\alpha\beta}\delta_{\beta\alpha}\right\} \quad ,$$

which finally equals

$$\sum_\alpha\left[\beta O_m\Lambda_-(-\vec{q})O_n^+\beta\Lambda_+(\vec{p})\right]_{\alpha\alpha} \quad ,$$

which by definition is

$$\mathrm{Tr}\left[\beta O_m\Lambda_-(-\vec{q})O_n^+\beta\Lambda_+(\vec{p})\right] \quad .$$

Putting all this together, we get

$$\sum_{\substack{\text{Final spins}\\ \text{of electron}\\ \text{and neutrino}}} |M|^2 = \sum_n \sum_m C_n^* C_m \left[\int \beta O_n f\right]^* \left[\int \beta O_m f\right] \mathrm{Tr}\left[\beta O_m \Lambda_-(-\vec{q}) O_n^+ \beta \Lambda_+(\vec{p})\right] \quad .$$

ALLOWED TRANSITIONS

This means two things: $f(x) = 1$, and the non-relativistic operators must be used.

I. Pure Scalar

$$O_m = I \qquad O_n = I = O_n^+$$

$$C_m = C_s \qquad C_n = C_s \qquad C_n^* C_m = |C_s|^2 \quad .$$

Since the nucleon is non-relativistic,

$$\int \psi_f^* \beta \psi_o dV \approx \int \psi_f^* I \psi_o dV \equiv M_F, \quad \text{The Fermi matrix element.}$$

We must carry out the trace.

$$\mathrm{Tr}\left[\beta \left(\frac{\beta\vec{\gamma}\cdot\vec{q}+E_\nu}{2E_\nu}\right) \beta \left(\frac{\beta\vec{\gamma}\cdot\vec{p}+\beta m+W}{2W}\right)\right] \quad .$$

Using the properties of the Dirac matrices, and their traces, this becomes (most of the terms have zero trace)

$$1 + \mathrm{Tr}\left(\frac{(\vec{\gamma}\cdot\vec{q})(\vec{\gamma}\cdot\vec{p})}{4WE_\nu}\right) = 1 + \sum_j \sum_k \mathrm{Tr}\,\frac{\gamma^j\gamma^k q^j p^k}{4E_\nu W}$$

$$= 1 + \sum_j \sum_k \frac{g^{jk} q^j p^k}{E_\nu W}$$

$$= 1 - \frac{\vec{p}\cdot\vec{q}}{E_\nu W} \quad .$$

The complete contribution from the pure scalar term is

$$\sum_{\substack{\text{Final}\\\text{spins}\\\text{of leptons}}} |M|^2_{\text{pure scalar}} = |C_S|^2\ |M_F|^2 \left(1 - \frac{p\cos\theta}{W}\right) \quad ,$$

where θ is the angle between the electron and the neutrino. Note that $\frac{p}{W} = \frac{v}{c}$. For this pure scalar case, the electron and neutrino tend to come off in opposite directions.

II. Pure Vector

$$O_m = \beta \qquad O_n = \beta = O_n^+ \quad .$$

Result:

$$\sum_{\substack{\text{Final spins}\\\text{of leptons}}} |M|^2_{\text{pure vector}} = |C_V|^2\ |M_F|^2 \left(1 + \frac{p\cos\theta}{W}\right) \quad .$$

III. Pure Tensor

There are two types of terms

$$\left(O_m = \sigma^i, \qquad O_n = \sigma^i\right)$$

and

$$\left(O_m = \sigma^i, \qquad O_n = \sigma^j\right) \quad .$$

Consider:

$$O_m = \sigma^i \qquad O_n = \sigma^i = O_n^+$$

$$\mathrm{Tr}\left[\beta\sigma^i\left(\frac{\beta\vec{\gamma}\cdot\vec{q} + E_\nu}{2E_\nu}\right)\sigma^i\beta\left(\frac{\beta\vec{\gamma}\cdot\vec{p}+\beta m+W}{2W}\right)\right]$$

$$= 1 + \mathrm{Tr}\left[\frac{\sigma^i(\vec{\gamma}\cdot\vec{q})\sigma^i(\vec{\gamma}\cdot\vec{p})}{4E_\nu W}\right]$$

$$= 1 + \frac{1}{4E_\nu W}\sum_{\ell}\sum_{m} \mathrm{Tr}(\sigma^i\gamma^\ell\sigma^i\gamma^m)q^\ell p^m$$

$$= 1 - \frac{1}{4E_\nu W}\sum_{\ell\neq i}\sum_{m} \mathrm{Tr}(\gamma^\ell\gamma^m)q^\ell p^m + \frac{1}{4E_\nu W}\sum_{m} \mathrm{Tr}(\gamma^i\gamma^m)q^i p^m$$

$$= 1 + \sum_{\ell\neq i} \frac{q^\ell p^\ell}{E_\nu W} - \frac{q^i p^i}{E_\nu W}$$

$$= 1 + \frac{\vec{q}\cdot\vec{p}}{E_\nu W} - \frac{2\, q^i p^i}{E_\nu W} \quad .$$

This term is multiplied by $|C_T|^2 |\int\psi_f^*\beta\sigma^i\psi_o|^2 \approx |\int\psi_f^*\sigma^i\psi_o|^2 |C_T|^2$. The sum of the three terms of this type (x, y, and z) is

$$|C_T|^2 \left[|M_{GT}|^2 \left(1 + \frac{\vec{q}\cdot\vec{p}}{E_\nu W}\right) - \frac{2}{E_\nu W} \left(\left|\int\sigma_x\right|^2 q^x p^x + \left|\int\sigma_y\right|^2 q^y p^y + \int|\sigma_z|^2 q^z p^z \right) \right]$$

where the Gamow-Teller matrix element is defined by

$$\vec{M}_{GT} = \int \psi_f^* \vec{\sigma} \psi_o \, dV$$

$$|M_{GT}|^2 = \left(\int \psi_f^* \vec{\sigma} \psi_o \, dV\right)^* \cdot \left(\int \psi_f^* \vec{\sigma} \psi_o \, dV\right)$$

$$= \sum_{i=1}^{3} \left| \int \psi_f^* \sigma^i \psi_o \, dV \right|^2 = \sum_{i=1}^{3} \left| \int \sigma^i \right|^2 \quad .$$

If all the terms in which $O_m = \sigma^i, O_n = \sigma^j = O_n^+$ are added on, the complete result for the tensor interaction is

$$\sum_{\substack{\text{lepton} \\ \text{spins}}} |M|^2_{\substack{\text{pure} \\ \text{tensor}}}$$

$$= |C_T|^2 \left\{ |M_{GT}|^2 \left(1 + \frac{\vec{q}\cdot\vec{p}}{E_\nu W}\right) - \frac{1}{E_\nu W} \sum_{i=1}^{3} \sum_{j=1}^{3} \left[\int \sigma_i\right]^* \left[\int \sigma_j\right] \left(p^i q^j + p^j q^i\right) \right\} \quad .$$

In this form, the fact that the angular distribution is independent of the coordinate system is clearly demonstrated.

If the initial nucleus is unpolarized and the final nuclear spin is summed over, then, since there is no preferred direction in space

$$\overline{\left|\int \sigma_x\right|^2} = \overline{\left|\int \sigma_y\right|^2} = \overline{\left|\int \sigma_z\right|^2} = \frac{1}{3} |M_{GT}|^2$$

$$\overline{\left[\int \sigma_x\right]^* \left[\int \sigma_y\right]} = 0, \text{ etc,}$$

where the bar denotes averaging over initial nuclear polarizations and summing over final nuclear polarizations.

$$\sum_{\substack{\text{lepton spins;}\\ \text{final nuclear spin;}\\ \text{average over initial}\\ \text{nuclear spin}}} |M|^2_{\substack{\text{pure}\\ \text{tensor}}} = |C_T|^2 |M_{GT}|^2 \left(1 + \frac{1}{3} \frac{\vec{p}\cdot\vec{q}}{E_\nu W}\right) .$$

This result seems to be more generally valid than the unpolarized case just considered. If the basic nuclear wave functions have definite J^2 and J_z, then the off-diagonal terms must be zero*.

IV. Pure Axial Vector

Similar to tensor

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*See Jackson, Treiman, and Wyld, Phys. Rev. 106, 517 (1957) for more on polarization.

$$\sum_{\substack{\text{lepton}\\ \text{spins}}} |M|^2_{\substack{\text{pure}\\ \text{axial}\\ \text{vector}}} =$$

$$= |C_A|^2 \left\{ |M_{GT}|^2 \left(1 - \frac{\vec{q}\cdot\vec{p}}{E_\nu W}\right) + \frac{1}{E_\nu W} \sum_{i=1}^{3} \sum_{j=1}^{3} \left[\int \sigma_i\right]^* \left[\int \sigma_j\right] \left(p^i q^j + q^i p^j\right) \right\}$$

$$\sum_{\substack{\text{lepton spins;}\\ \text{final nuclear spin;}\\ \text{average over initial}\\ \text{nuclear spin}}} |M|^2_{\substack{\text{pure}\\ \text{axial}\\ \text{vector}}} = |C_A|^2 |M_{GT}|^2 \left(1 - \frac{1}{3} \frac{\vec{p}\cdot\vec{q}}{E_\nu W}\right) .$$

INTERFERENCE

V. Scalar-Vector

$$O_m = I \qquad O_n = \beta = O_n^+$$

$$\mathrm{Tr} \left[\beta \left(\frac{\beta\vec{\gamma}\cdot\vec{q} + E_\nu}{2E_\nu} \right) \left(\frac{\beta\vec{\gamma}\cdot\vec{p} + \beta m + W}{2W} \right) \right] = \frac{m}{W} .$$

Adding the case where I and β are interchanged gives another similar term.

$$\sum_{\substack{\text{lepton}\\ \text{spins}}} |M|^2_{\substack{\text{scalar-}\\ \text{vector}}} = \left(C_S^* C_V + C_V^* C_S\right) |M_F|^2 \frac{1}{W} ,$$

since W is measured in units of mc^2.

VI. Scalar-Tensor Interference

$$O_m = I \qquad\qquad O_n = \sigma_\ell = O_n^+$$

$$\mathrm{Tr}\left[\beta\left(\frac{\beta\vec{\gamma}\cdot\vec{q}+E_\nu}{2E_\nu}\right)\sigma_\ell\beta\left(\frac{\beta\vec{\gamma}\cdot\vec{p}+\beta m+W}{2W}\right)\right]$$

$$= \frac{\mathrm{Tr}(\vec{\gamma}\cdot\vec{q})\sigma_\ell(\vec{\gamma}\cdot\vec{p})}{4E_\nu W}$$

$$= \frac{1}{4E_\nu W}\sum_r\sum_s \mathrm{Tr}\left(\gamma^r\sigma^\ell\gamma^s\right)q^r p^s$$

$$= \frac{i}{4E_\nu W}\sum_r\sum_s \mathrm{Tr}\left(\gamma^r\gamma^j\gamma^k\gamma^s\right)q^r p^s \qquad j,k,\ell \text{ cyclic} \quad (j\neq k)$$

$$= \frac{i}{E_\nu W}\sum_r\sum_s \left(g^{rj}g^{ks}-g^{rk}g^{js}\right)q^r p^s$$

$$= \frac{i}{E_\nu W}\left(q^j p^k - q^k p^j\right) \qquad j,k,\ell \text{ cyclic} \quad .$$

This complete term is

$$\frac{i}{E_\nu W}C_S C_T^* M_F\left[\int\sigma^\ell\right]^*\left(q^j p^k - q^k p^j\right) \quad .$$

After summing over the three components of spin, we obtain

$$\frac{i}{E_\nu W} C_S C_T^* M_F \left[\int \vec{\sigma}\right]^* \cdot (\vec{q} \times \vec{p}) \quad .$$

If we include the case when O_m and O_n are interchanged, the result is

$$\sum_{\substack{\text{lepton}\\\text{spins}}} |M|^2_{\substack{\text{scalar-}\\\text{tensor}}} = 2\,\mathrm{Re}\left\{ \frac{i}{E_\nu W} C_S C_T^* M_F \left[\int \vec{\sigma}\right]^* \cdot (\vec{q} \times \vec{p}) \right\} \quad .$$

If there is no polarization, and final nuclear spins are summed over, this term will be zero. If time reversal is satisfied, this term will be zero. The Vector-Axial Vector Interference term is the negative of the Scalar-Tensor term. The other Fermi-Gamow-Teller interferences are identically zero.

VII. Tensor-Axial-Vector Interference

$$O_m = \sigma_i \qquad\qquad O_n = \beta\sigma_j = O_n^+$$

$$\mathrm{Tr}\left[\beta\sigma_i \left(\frac{\beta\vec{\gamma}\cdot\vec{q}+E_\nu}{2E_\nu}\right) \sigma_j \left(\frac{\beta\vec{\gamma}\cdot\vec{p}+\beta m+W}{2W}\right)\right] = \frac{m}{W}\,\delta_{ij} \quad .$$

Adding together all terms of this type, as well as the ones in which the role of tensor and axial-vector is interchanged, gives

$$\sum_{\substack{\text{lepton}\\\text{spins}}} |M|^2_{\substack{\text{tensor-}\\\text{axial}\\\text{vector}}} = \left(C_T C_A^* + C_A C_T^*\right) |M_{GT}|^2 \frac{1}{W} \quad .$$

Adding together all terms considered produces the formula given on pages 279-280 of the Rose article and reproduced below. The probability per unit time that an electron and neutrino are emitted into the relative solid angle $d\Omega = 2\pi \sin\theta \, d\theta$, with the electron having energy between W and W + dW, is (for the allowed transitions)

$$N_{\mp}(W,\theta)dWd\Omega = \left(\frac{mc^2}{\hbar}\right)\frac{1}{4\pi^3} F(\pm Z,W) \sqrt{W^2-1}\, W(W_o-W)^2 \times$$

$$\times \left[\left(|C_S'|^2 + |C_V'|^2\right)|M_F|^2 + \left(|C_T'|^2 + |C_A'|^2\right)|M_{GT}|^2\right] \times$$

$$\times \left(1 + \frac{ap\cos\theta}{W} \pm \frac{b}{W}\right) dW \sin\theta \, d\theta \quad .$$

The original coupling constants (unprimed) have the dimensions energy $\times$ volume. The primed coupling constants in this expression are dimensionless numbers equal to the unprimed constants divided by the natural units of energy and volume, e.g., $C_S' = C_S \Big/ (mc^2)\left(\frac{\hbar}{mc}\right)^3$. Now drop the primes.

The upper and lower signs refer to negative and positive (charge) electrons, respectively. The main Coulomb effect, the factor $F(\pm Z,W)$ has been written in, even though it hasn't been considered explicitly thus far. All energies are measured in units of mc^2. One factor of 2π has come in from the $\frac{2\pi}{\hbar}$ in the perturbation theory formula. Another factor of 2π comes from $d\Omega = 2\pi \sin\theta \, d\theta$, and a factor of 4π comes from integrating over the electron's directions (only the relative electron-neutrino angle remains).

$$a = \frac{\frac{1}{3}\left(|C_T|^2 - |C_A|^2\right)|M_{GT}|^2 - \left(|C_S|^2 - |C_V|^2\right)|M_F|^2}{\left(|C_T|^2 + |C_A|^2\right)|M_{GT}|^2 + \left(|C_S|^2 + |C_V|^2\right)|M_F|^2}$$

$$b = \gamma \frac{\left(C_S^* C_V + C_V^* C_S\right)|M_F|^2 + \left(C_T^* C_A + C_A^* C_T\right)|M_{GT}|^2}{\left(|C_S|^2 + |C_V|^2\right)|M_F|^2 + \left(|C_T|^2 + |C_A|^2\right)|M_{GT}|^2}$$

$$\gamma = \sqrt{1 - (\alpha Z)^2} \quad .$$

The term $\pm b/W$ is the Fierz interference term. It does not appear in a theory which treats electrons and positrons symmetrically (in the absence of coulomb effects). All terms arising from Fermi-Gamow-Teller interference have been omitted from the above expression. The Gamow-Teller part of the coefficient (a), has been written for the no-polarization case.

If one integrates the energy-angle transition probability over the electron-neutrino relative angle, the term in cos θ contributes nothing, and the result is the allowed spectrum

$$N_{\mp}(W)dW = \frac{mc^2}{\hbar}\frac{1}{2\pi^3} F(\pm Z,W)W\sqrt{W^2-1}\,(W_o-W)^2$$
$$\times\left[\left(|C_S|^2 + |C_V|^2\right)|M_F|^2 + \left(|C_T|^2 + |C_A|^2\right)|M_{GT}|^2\right]\left(1 \pm \frac{b}{W}\right)dW \quad .$$

Except for the interference term b/W, the spectrum shape is independent of the coupling.

FORBIDDEN TRANSITIONS

Two effects were neglected in the allowed transitions: The relativistic operators $\vec{\alpha}$, $\beta\vec{\alpha}$, γ_5 and $\beta\gamma_5$; and retardation, i.e., the higher terms in the expansion of $e^{-i(\vec{p}+\vec{q})\cdot\vec{r}}$. This latter expansion can be written

$$e^{-i(\vec{p}+\vec{q})\cdot\vec{r}} = \sum_{n=0}^{\infty} \frac{1}{n!} (-i)^n \left[(\vec{p}+\vec{q})\cdot\vec{r}\right]^n \quad .$$

The n^{th} term in this expansion has angular momentum components n, n-2, n-4,... $\binom{1}{0}$. The reason every other angular momentum occurs is that the n^{th} term has definite parity $(-1)^n$. The general selection rules for an arbitrary transition can be obtained by combining these facts with the intrinsic properties of the operators. $\vec{\sigma}$ and $\vec{\alpha}$, being vectors, carry one unit of angular momentum along with them; β, I, and γ_5 have no angular momentum. $\vec{\sigma}$, β, and I do not change the parity of a wave function, whereas $\vec{\alpha}$ and γ_5 do. For the allowed transitions, the result is no parity change for Fermi and Gamow-Teller, $\Delta J = 0$ for Fermi, and $\Delta J = \pm 1,0$ for Gamow-Teller (no $0 \rightarrow 0$, however).

The first forbidden transitions arise from two types of terms: the non-relativistic operators multiplied by $\left[-i(\vec{p} + \vec{q}) \cdot \vec{r}\right]$ and the relativistic operators multiplied by unity. All first forbidden transitions have the selection rule that the parity must change; in fact, for the n^{th} forbidden transitions the parity change is $(-1)^n$. The angular momentum selection rules follow directly from the above remarks except in that part of the tensor and axial vector interactions which involves a product of a spin component σ_i with a position component x_j.*

These nine terms $\sigma_i x_j$ form a tensor which can be broken down into three parts: (1) a scalar, $\vec{\sigma}\cdot\vec{r}$ (one component); (2) a vector $\vec{\sigma} \times \vec{r}$ (three components); and (3) a symmetric traceless tensor (five independent components). The scalar produces no change in angular momentum; the vector has the selection rule $\Delta J = \pm 1, 0$ (no $0 \rightarrow 0$); and the symmetric, traceless tensor, which carries two units of angular momentum, has the rule $\Delta J = \pm 2, \pm 1, 0$ (no $0 \rightarrow 0$, $\frac{1}{2} \rightarrow \frac{1}{2}$, $0 \longleftrightarrow 1$). The exceptions follow directly from the vector model rule for the addition of angular momenta.

In each order of forbiddenness n, there will be a tensor coming from the Gamow-Teller coupling, which carries n+1 units of angular momentum This is the maximum possible angular momentum change in an n^{th} forbidden transition. This is called the "unique" forbidden transition.

One calculates the electron-neutrino energy-angle distribution in the same way as for allowed transitions. In the absence of coulomb effects, the lepton portion of the matrix element separates from the nuclear part. The sum over lepton spins proceeds as before; there are

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*E. Konopinski 1955, Beta and Gamma Ray Spectroscopy, ed. K. Siegbhan, Chap. X, p. 294.

some new traces to be evaluated, however, coming from the relativistic operators $\vec{\alpha}$, γ_5. The traces involving $\vec{\alpha}$ are similar to those involving $\vec{\sigma}$, and the traces involving γ_5 are similar to those which contain the identity. Additional angular dependencies arise from the factors of $\vec{p} + \vec{q}$.

For the n-times forbidden spectrum integrated over all angles, one writes

$$N(W)dW = \left(\frac{mc^2}{\hbar}\right) \frac{1}{2\pi^3} F(Z,W) \sqrt{W^2-1} \; W(W_o-W)^2 S_n(W)dW \quad ,$$

where $S_n(W)$ is called the shape factor. The shape factors are quite involved and depend upon the coulomb effect rather strongly.

$$S_o(W) = \left[\left(|C_S|^2 + |C_V|^2\right) |M_F|^2 + \left(|C_T|^2 + |C_A|^2\right) |M_{GT}|^2\right] \left(1 \pm \frac{b}{W}\right) \quad .$$

EXPERIMENTAL SITUATION IN BETA DECAY (before parity)

Experimental spectra are generally displayed in the form of Kurie plots:

$$\left(N(W) \Big/ F(Z,W) \, W\sqrt{W^2-1}\right)^{\frac{1}{2}} \quad \text{versus } W.$$

For the most carefully measured allowed spectra, Cu^{64}, N^{13}, S^{35}, this plot is a straight line, which indicates that the Fierz interference term is negligible. A conservative upper limit on the coefficient 'b' is $b < 0.2$. Since b involves products of the two Fermi coupling constants, and products of the two Gamow-Teller constants, the assumption was made that only one of the Fermi couplings and only one of the Gamow-Teller couplings is present.*

The conclusion that Fermi and Gamow-Teller couplings are both necessary followed from the existence of allowed transitions such as $He^6 \to Li^6$ which has $\Delta J = 1$. This is allowed under Gamow-Teller, but forbidden by Fermi selection rules. Similarly $Cl^{34} \to S^{34}$ is an allowed $0 \to 0$ transition which can take place under Fermi, but is forbidden under Gamow-Teller selection rules. From a study of several decays (with some reasonable approximations for the nuclear matrix elements) the conclusion was reached that the Fermi and Gamow-Teller couplings are approximately equal in strength.

The decision that the correct Gamow-Teller coupling is the tensor, and not the axial-vector, resulted from the electron-nuclear recoil angular correlation experiment in the decay of He^6. Using conservation of linear momentum, one can convert the expressions above, which are written in terms of the electron-neutrino angle, to expressions involving the electron-nucleus angle. The results of this difficult experiment agree rather well with the prediction of the tensor coupling, and disagree with axial-vector coupling. The shapes of once forbidden spectra give information about the proper choice of Fermi coupling. These are more difficult to

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*C. S. Wu 1955, Beta and Gamma Ray Spectroscopy, ed. K. Siegbhan, Chap. XI.

unravel than allowed spectra and indicated that the scalar coupling is the correct one.

There seems to be some evidence that the pseudoscalar coupling constant is not zero.

An approximate value for the Fermi coupling constant is $C_F = 1.41 \times 10^{-49}$ ergs-cm^3. This means that the dimensionless Fermi coupling constant is $C_F' \simeq 3.00 \times 10^{-12}$. This shows how weak the β-decay coupling is. Using the proton-mass instead of the electron as the unit of mass gives $C_F''(M_P) = 1.01 \times 10^{-5}$.

SOME ADDITIONAL PROPERTIES OF THE GAMMA MATRICES

(same definitions as in K. Ford's lectures)

Greek indices μ, ν,... go from zero to three, and Roman indices j,k,... go from one to three (x to z)

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \qquad \gamma^0 = \beta = \gamma^{0+} \qquad \gamma^k = \beta\alpha^k = -\gamma^{k+} \qquad \gamma^\mu\gamma^{\mu+} = I$$

$g^{\mu\nu}$ =

μ \ ν	0	1	2	3
0	1	0	0	0
1	0	-1	0	0
2	0	0	-1	0
3	0	0	0	-1

$$\gamma_5 = -i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \gamma^0\gamma^1\gamma^2\gamma^3$$

I. Relation between γ's and σ's

$$\gamma^j\gamma^k = -\sigma^j\sigma^k$$

$$\gamma_5\gamma^k = i\beta\sigma^k$$

$$\gamma^j\gamma^k = -i\sigma^\ell \qquad (j,k,\ell \text{ cyclic permutation of } x,y,z)$$

$$\sigma^k\gamma^j = -\gamma^j\sigma^k \qquad (j \neq k)$$

$$\sigma^k\gamma^k = \gamma^k\sigma^k$$

$$\sigma^k\beta = \beta\sigma^k$$

$$\sigma^j\gamma^k = i\gamma^\ell \qquad (j,k,\ell \text{ cyclic})$$

II. Traces*

$\mathrm{Tr}(A) = \sum_\mu A_{\mu\mu}$ = sum of the diagonal elements of A. The trace of the product of an odd number of γ matrices vanishes.

In particular:

$$\mathrm{Tr}(\gamma^\mu) = 0$$

$$\mathrm{Tr}(\sigma^k\gamma^\mu) = 0 \qquad (\text{a } \sigma \text{ is made up of two } \gamma\text{'s})$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = \mathrm{Tr}(\gamma^\nu\gamma^\mu) = \tfrac{1}{2}\,\mathrm{Tr}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) = g^{\mu\nu}\mathrm{Tr}(I) = 4g^{\mu\nu} \quad .$$

$$\mathrm{Tr}(\gamma^\mu\gamma^{\nu+}) = 4\delta^{\mu\nu}$$

In particular:

$$\mathrm{Tr}(\beta^2) = 4$$

$$\mathrm{Tr}(\gamma^{k2}) = -4$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 0 \qquad (\mu \neq \nu)$$

$$\mathrm{Tr}(\sigma^k) = 0$$

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*Schweber, Bethe, and de Hoffmann 1955, Mesons and Fields I, Chapter I, section 7d.

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4g^{\mu\nu}g^{\rho\sigma} - 4g^{\mu\rho}g^{\nu\sigma} + 4g^{\mu\sigma}\ g^{\nu\rho} \quad .$$

$$\mathrm{Tr}(\gamma_5) = 0 \quad .$$